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Scaling properties of antipercolation hulls on the triangular lattice

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Abstract. The critical properties of the perimeters (or 'hulls') of antipercolation clusters are studied in two dimensions by Monte Carlo simulations on the triangular lattice. Two different types of hulls are constructed with the help of two kinetic walk algorithms. For the standard hull, we very accurately determine the size distribution exponent $\tau' = 2.1425 \pm 0.0003$, as well as the fractal dimension $d_f^S = 1.750 \pm 0.001$. The corresponding exponents for regular percolation hulls (15/7 and 7/4 respectively) are within the error bars of our results for antipercolation. For the reduced hull, we obtain the estimate $d_f^R = 1.334 \pm 0.004$ for the fractal dimension, a result which is again close to that found for regular percolation.

1. Introduction

Antipercolation (or AB percolation) is a variant of regular percolation in which the connectivity rules are modified (Mai and Halley 1980, Sevšek *et al* 1983). In both cases, the lattice sites are coloured black with probability p or white with probability $1-p$, but in antipercolation two first-neighbour sites are connected only if their colours are different. Antipercolation clusters are therefore made up of alternating black and white sites and they are surrounded by a double layer of sites which all have the same colour. Applications of the model include the propagation of venereal epidemics (Wu and Bradley 1991a). In two dimensions (2D), the triangular lattice is the only simple planar lattice where a transition occurs (Appel and Wierman 1987, Wierman and Appel 1987). Numerical studies of this transition indicate that the bulk critical exponents are the same as for regular percolation (Sevšek *et al* 1983, Nakanishi 1987, Wu and Bradley 1991b).

The interest in the structure of percolation clusters has recently broadened to include the cluster perimeter (or 'hull') in two dimensions (for a review, see Ziff (1989)) as well as in three dimensions (Strenski *et al* 1991, Bradley *et al* 1991). The relation

$$d_f^S = 1 + 1/\nu = \tfrac{7}{4}$$

between the fractal dimension d_f^S of the hull and the correlation length exponent ν for percolation clusters in 2D was first proposed by Sapoval *et al* (1985). This conjecture was subsequently supported by studies of diffusion fronts (Bunde and Gouyet 1985)

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and was ultimately proven to be exact by Saleur and Duplantier (1987). Additional scaling relations have been established which interrelate the other hull critical exponents (Ziff 1986, Strenski *et al* 1991, Bradley *et al* 1991). An analogy between the hull and a self-avoiding walk at the Θ' point was drawn (Coniglio et al 1987, Duplantier and Saleur 1987), and it was argued that the reduced hull (Grossman and Aharony 1986) must have a fractal dimension $d_{\mathrm{f}}^{\mathrm{R}} = \frac{4}{3}$, since the unstable Θ' tricritical point is affected by the resulting restrictions on the hull configurations (Saleur and Duplantier 1987).

The derivation of the result $d_{\mathrm{f}}^{\mathrm{S}} = \frac{7}{4}$ makes use of the equivalence between percolation and the $q = 1$ Potts model. For antipercolation, mappings to modified $q = 1$ Potts models have been proposed (Turban 1983, Halley 1983, Wu and Bradley 1991a), but the Coulomb gas mapping used in the case of percolation cannot be readily extended to these Potts models. As a result, there are no exact results available for antipercolation. Moreover, the hull exponents have never been estimated for this model.

The surface of an antipercolation cluster has some unusual features as compared to regular percolation. For instance, it is possible that first-neighbour sites belong to distinct clusters, even if they are of the same colour. This is in contrast with regular percolation where two clusters cannot be closer than a second-neighbour distance. Moreover, since the sites at the perimeter of an antipercolation cluster all have the same colour, they are not directly connected to each other. They are instead connected by internal sites of the opposite colour. It seemed worthwhile to us to test whether these unusual features affect the critical properties of the hull.

2. Algorithm

In 2D, it is well known that hulls of percolation clusters can be constructed by a kinetic random walk which follows a set of appropriate rules (Ziff *et al* 1984, Weinrib and Trugman 1985, Gouyet 1988). These rules must be modified, according to the nature of the underlying lattice and to the specific case considered (e.g., bond percolation, site percolation, etc). For antipercolation, we start with a triangular lattice which is completely uncoloured, except for a pair of white first-neighbour sites. The walk associated with the hull is constructed on the honeycomb lattice, the dual to the original lattice. Initially, the walk consists of a single step bisecting the bond between the two white sites and is oriented in one of the two possible directions. The closest site of the triangular lattice in the forward direction of the walk is called the target site (figure 1(a)). At each Monte Carlo step, the occupation of the target site is checked. If this site is uncoloured, it is coloured black with probability p or white with probability $1 - p$. A previously coloured target site is unaltered. If the target site is white, the walk turns right through a white–white bond (figure 1(b)). Conversely, if it is black, the walk backtracks since it must not cross a white–black bond (figure 1(c)). This procedure is repeated until the walk closes or until the number of steps is greater than a given cut-off value (see figure 2(a)). With the above rules, the cluster always lies on the right-hand side of the walker, so that a walk closing clockwise (anticlockwise) is an external (internal) hull. This type of walk is not self-avoiding since it often has to backtrack. However, there is a type of steric hindrance in the model since each bond can be visited a maximum of two times.

In practice, the simulations were performed on a $65\,536 \times 65\,536$ lattice, using a data blocking scheme (Ziff *et al* 1984). The cut-off was usually set to $2^{20} - 1$, but it

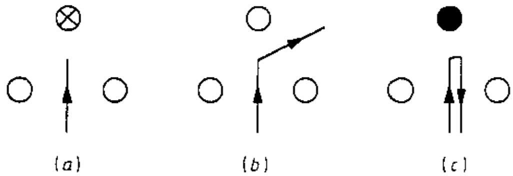

Figure 1. The kinetic walk algorithm used to construct the standard hull. (*a*) An elementary step of the kinetic walk (|) bisects two white sites (○) of the triangular lattice and points towards the target site (⊗). (*b*) If the target site is white, the walk turns right. (*c*) If the target site is black, the walk backtracks.

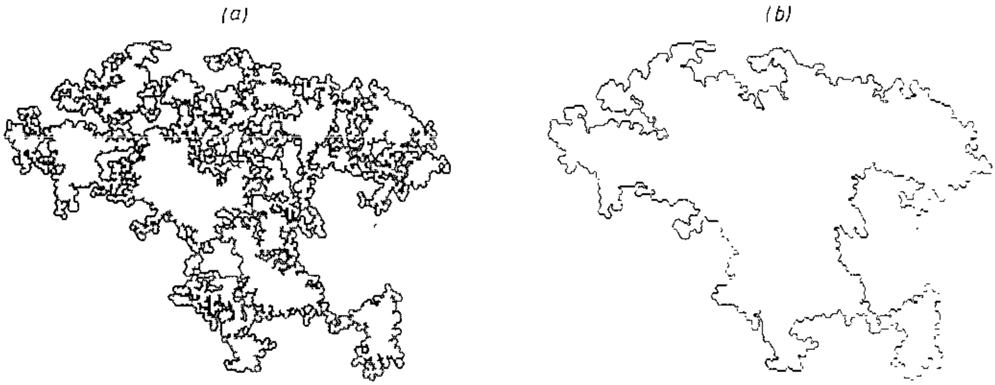

Figure 2. The external perimeter of an antipercolation cluster on the triangular lattice. (*a*) The standard hull contains 24 722 steps. (*b*) The corresponding reduced hull contains only 3416 steps.

was occasionally increased to $2^{22}-1$. All the walks generated during the simulations remained within the lattice boundaries so that no finite-size bias can be attributed to the lattice dimensions.

3. Results for the standard hull

We calculated both the number $N_{\mathrm{ext}}(s)$ of external and the number $N_{\mathrm{int}}(s)$ of internal perimeters which closed after exactly s steps. These numbers are very sensitive to the value of the probability p: below the percolation threshold p_{c} there are more external perimeters than internal ones, while the opposite is the case above p_{c}. It was first suggested by Ziff (1986) that, at $p = p_{\mathrm{c}}$, the ratio $N_{\mathrm{int}}(s)/N_{\mathrm{ext}}(s)$ tends to one as s tends to infinity. To illustrate this point, we assume the usual scaling form (Stauffer 1979, Ziff 1986)

$$N_{\alpha}^{\delta}(s) \sim s^{1-\tau'} f_{\alpha}^{\delta}(|p-p_{\mathrm{c}}|s^{\sigma'}) \tag{1}$$

where the superscript δ refers to the regions above ($\delta = +$) and below ($\delta = -$) p_c and the subscript α distinguishes internal (α = int) from external (α = ext) perimeters. For a given number of steps s, the ratio of internal to external perimeters is given by

$$R^{\delta}(x) = \frac{N^{\delta}_{\mathrm{int}}(s)}{N^{\delta}_{\mathrm{ext}}(s)} = \frac{f^{\delta}_{\mathrm{int}}(x)}{f^{\delta}_{\mathrm{ext}}(x)} \tag{2}$$

where $x \equiv |p - p_c|s^{\sigma'}$. Clearly $R^{+}(x) \geq 1$, whereas $R^{-}(x) \leq 1$. At $p = p_c$, we have $N^{+}_{\alpha} = N^{-}_{\alpha}$ and, as a consequence, $f^{+}_{\alpha}(0) = f^{-}_{\alpha}(0)$, so that finally $R^{+}(0) = R^{-}(0) = 1$. We have considered here the general case where $f^{\delta}_{\mathrm{int}} \neq f^{-\delta}_{\mathrm{ext}}$. However, for site percolation on the triangular lattice, the situation is naturally symmetrical about $p_c = \frac{1}{2}$, and so $f^{\delta}_{\mathrm{int}}$ and $f^{-\delta}_{\mathrm{ext}}$ coincide. Moreover, since there is numerical evidence that this equality is also valid for site percolation on the square lattice (Ziff *et al* 1984), we believe this could be a general result.

To avoid large fluctuations as well as finite-size effects, we calculated the ratio

$$\rho^{\delta}(p) = \sum_{s=s_1}^{s=s_2} N^{\delta}_{\mathrm{int}}(s) \left(\sum_{s=s_1}^{s=s_2} N^{\delta}_{\mathrm{ext}}(s) \right)^{-1} = \sum_{s=s_1}^{s=s_2} s^{1-\tau'} f^{\delta}_{\mathrm{int}}(x) \left(\sum_{s=s_1}^{s=s_2} s^{1-\tau'} f^{\delta}_{\mathrm{ext}}(x) \right)^{-1} \tag{3}$$

with the summation limits $s_1 = 2^{16}$ and $s_2 = 2^{20} - 1$ (table 1), rather than computing R^{δ}. The critical probability p_c is given by the value of p at which $\rho^{\delta}(p) = 1$ and, from table 1, we obtain the estimate $p_c = 0.215\,65 \pm 0.000\,03$. The error here is purely statistical (Ziff 1986). This result is slightly larger than a previous estimate, $p_c = 0.215\,24 \pm 0.000\,34$ (Nakanishi 1987). Our estimate of p_c is probably slightly below the exact value, since when we increased the cut-off from $2^{20} - 1$ to $2^{22} - 1$ steps, $\rho(0.215\,65)$ decreased from 0.98 to 0.95. Using our assumption that $f^{\delta}_{\mathrm{int}} = f^{-\delta}_{\mathrm{ext}}$, equation (3) gives

$$\rho^{+}(p_c + \Delta p) = 1/\rho^{-}(p_c - \Delta p) \tag{4}$$

for small $\Delta p > 0$. Within the error bars, this last equality agrees with our numerical results (table 1). This lends additional support to the idea that the scaling functions are symmetrical about p_c.

Table 1. The ratio ρ of the number of internal to external perimeters for different p values. We summed all the perimeters with a number of steps in the range $[2^{16}, 2^{20}]$. Altogether, 20 000 perimeters (of all sizes) were constructed for each of the p values, except for $p = 0.21565$, where this amount was increased to 150 000. The errors quoted here are statistical in origin. The value $p = 0.215\,65$ for which ρ is the closest to unity gives an accurate estimate of the critical probability.

p	ρ
0.2154	0.62 ± 0.05
0.2155	0.75 ± 0.06
0.2156	0.95 ± 0.07
0.21565	0.98 ± 0.03
0.2157	1.10 ± 0.09
0.2158	1.31 ± 0.10
0.2159	1.58 ± 0.12

According to equation (1), the fraction $F(s)$ of perimeters which are longer than s steps must be proportional to $s^{2-\tau'}$ at p_c. To determine the distribution exponent τ', we made a log–log plot of $F(s)$ against s (figure 3). A linear fit for s in the range $[2^7, 2^{20}]$ gives $\tau' = 2.1425 \pm 0.0003$, in excellent agreement with the value $\tau' = 15/7 \simeq 2.1428...$ which is believed to be exact for regular percolation (Ziff 1986, Saleur and Duplantier 1987). We also estimated the fractal dimension d_f^S of the perimeters at $p = p_c = 0.21565$. We calculated the average square distance $R^2(s)$ between two sites on the hull separated by $s = 2^6, 2^7, ..., 2^{19}$ steps. The data were averaged over the 6930 perimeters which remained open after $2^{20} - 1$ steps (out of a total of 50000 perimeters). The plot of $\log_2 s$ as a function of $\log_2 R$ gives a very straight line, with slope $d_f^S = 1.750 \pm 0.0001$, for s in the range $[2^9, 2^{19}]$ (figure 4). Here again our estimate is, within the error bars, equal to its exact counterpart for regular percolation, $d_f^S = \frac{7}{4}$ (Sapoval *et al* 1985, Saleur and Duplantier 1987).

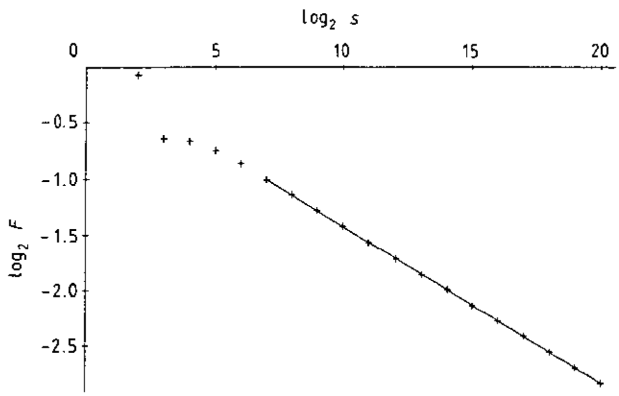

Figure 3. $\log_2 F$ as a function of $\log_2 s$. The data points are indicated with crosses. The slope given by a linear least squares fit for $s \geq 2^7$ (full line) is equal to -0.1425 ± 0.0003.

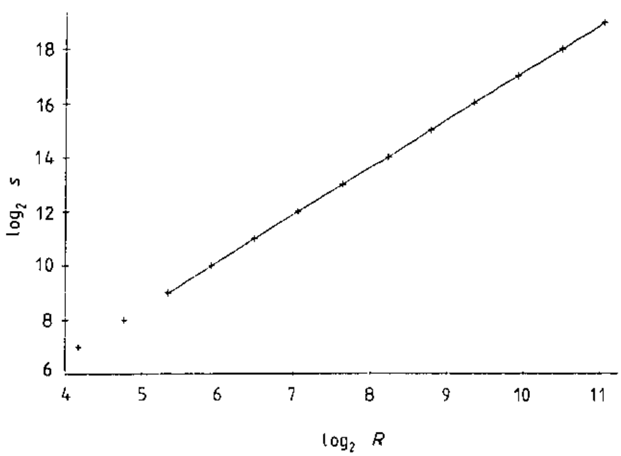

Figure 4. $\log_2 s$ as a function of $\log_2 R$. The full line is a linear fit of the data points with s in the range $[2^9, 2^{19}]$. Its slope gives the fractal dimension $d_f^S = 1.750 \pm 0.0001$.

Finally, we tested the scaling behaviour of the perimeter distribution around p_c by calculating the scaling functions defined in equation (1),

$$f_\alpha^\delta(x) = s^{\tau'-1} N_\alpha^\delta(s). \tag{5}$$

For regular percolation, the exponent σ' is connected to the correlation-length exponent ν by

$$\sigma' = 1/(1+\nu) \tag{6}$$

and is equal to $\frac{3}{7}$ in 2D (Ziff 1986). If we assume this relation to be valid for antipercolation as well, then σ' must be unchanged, since ν has the same value in both cases (Sevšek *et al* 1983). We thus used the value $\sigma' = \frac{3}{7}$ to compute the scaling functions given in equation (5). The excellent collapse of the numerical data which is observed in figure 5 justifies this choice *a posteriori*. This plot also confirms the symmetry of the scaling functions, since the data points for $f_{\rm int}^{\delta}$ and $f_{\rm ext}^{-\delta}$ fall on the same curve. We also observe that the two scaling functions have the same asymptotic behaviour for $x \to 0$ and $x \to \infty$, but that they differ substantially in the intermediate region. Finally, both functions tend to the same constant value when $x \to 0$, as expected.

4. Results for the reduced hull

It has been shown recently that for continuum percolation in 2D, the standard hull with fractal dimension $d_{\rm f}^{\rm S} = \frac{7}{4}$ eventually crosses over a reduced hull with fractal dimension $d_{\rm f}^{\rm R} = \frac{4}{3}$ if the characteristic length used to construct the hull is increased (Rosso 1989, Kolb 1990). This confirms and generalizes the results obtained for regular percolation on the square and triangular lattices (Grossman and Aharony 1986, 1987, Freund and Grassberger 1991). It is also tempting to test this prediction in the case of antipercolation. To obtain the reduced hull, we first used the algorithm described in section 2 to construct an *external* standard hull (internal hulls were discarded). Then, we discarded all but the outermost part of the standard hull by deleting all the loops which were connected to it by twice-occupied bonds (see figure 2). This was achieved with the help of a new type of kinetic walk which was started at an arbitrary point along the external edge of the hull. The rules for this new walk are similar to those given in section 2 (figure 1), but, since the new walk traces out the outermost part of an external perimeter, the target site is always white. The rules given in figure 1 must be modified as follows: if the target site has a black first-neighbour then turn left; otherwise turn right. With these new rules, the walk is no longer allowed to cut a bond between two white sites belonging to the antipercolation cluster. For $p = p_c$, we constructed a total of 54 700 reduced hulls. We calculated their radius of gyration as a function of the number of steps s and binned the data over increasing ranges of s values. Let $R_{\rm g}(s_i)$ be the average radius for all the hulls with s in the range $[s_i, \sqrt{2}s_i]$, where $s_i = 2^{(i+4)/2}$ $(i = 1, 2, \ldots)$. In figure 6, a log–log plot of s_i as a function of $R_{\rm g}(s_i)$ is shown. Since the cut-off was imposed upon the original standard hulls rather than on the corresponding reduced hulls, the data for large s_i values are biased. The two points furthest to the right in figure 6 are affected most by this bias and have therefore been discarded in the following analysis. The reduced hulls are much smaller than the standard ones, and so the log–log plot in figure 6 shows a systematic curvature. A

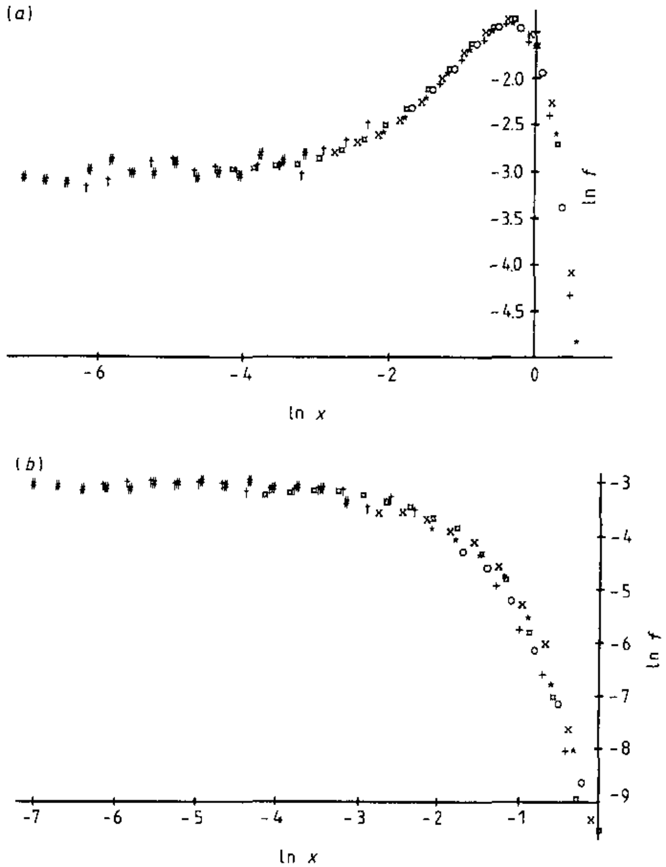

Figure 5. The scaling functions for the size distribution of perimeters longer than 62 steps. (*a*) The function $f_{\rm ext}^{-}$ is plotted for $p = 0.170$ (+), 0.185 (○), 0.195 (∗), 0.205 (×) and 0.213 (o). The function $f_{\rm int}^{+}$ is plotted for $p = 0.2158$ (‡) and 0.2160 (†). (*b*) The functions $f_{\rm int}^{-}$ and $f_{\rm ext}^{+}$. The p values and the symbols used are the same as in (*a*).

finite-size scaling analysis is thus necessary to determine $d_{\rm f}^{\rm R}$ accurately. We therefore calculated the standard finite-size estimators

$$d_2^{\rm R}(i) = \frac{\log(s_{i+2}/s_{i-2})}{\log[R_{\rm g}(s_{i+2})/R_{\rm g}(s_{i-2})]} \tag{7}$$

and plotted them as a function of $1/R_{\rm g}(s_i)$. After a crossover regime, the estimators for the largest hulls are well fit by a straight line, indicating that the corrections to scaling have a simple form (figure 7). A linear extrapolation to $1/R_{\rm g} = 0$ gives the estimate $d_{\rm f}^{\rm R} = 1.334 \pm 0.004$, in excellent agreement with the value $\frac{4}{3}$ obtained for regular percolation.

5. Summary

We have constructed the standard hulls of antipercolation clusters on the triangular lattice. We have shown that the ratio ρ of the number of internal hulls to the number

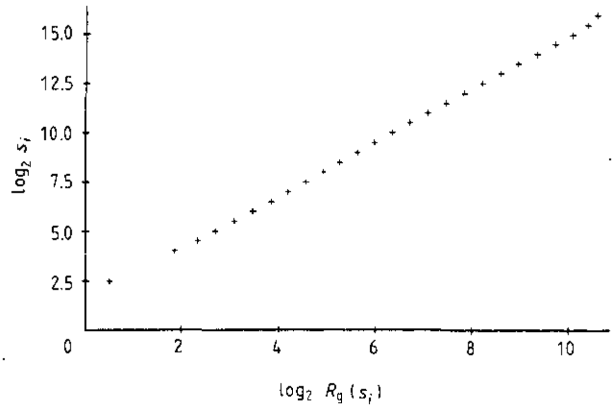

Figure 6. $\log_2 s_i$ as a function of $\log_2 R_g(s_i)$. A curvature is visible, even for large s_i values.

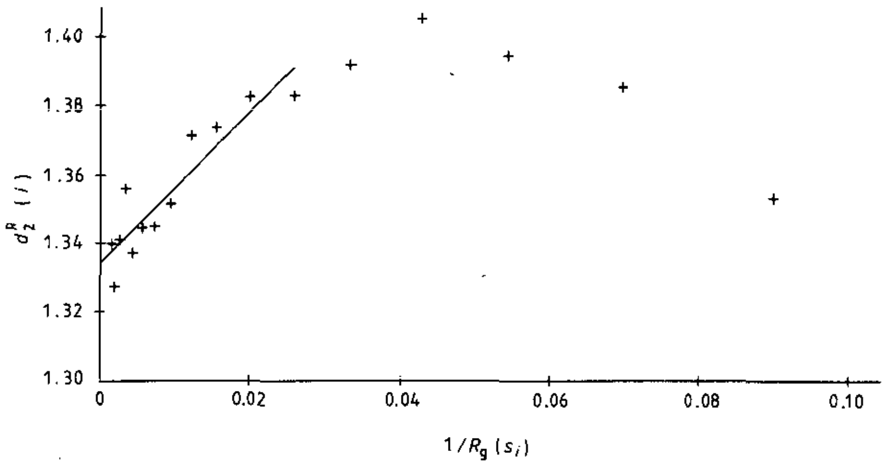

Figure 7. The finite-size estimator $d_2^R(i)$ plotted as a function of the inverse of the average radius of gyration $1/R_g(s_i)$ (crosses). A crossover regime is observed for the small hulls. The full line is a linear fit for $s_i > 90$ giving $d_f^R = 1.333 \pm 0.004$ at the intercept.

of external hulls must be equal to one when $p = p_c$. Using this criterion, we obtained the estimate $p_c = 0.215\,65 \pm 0.000\,03$. Our numerical results show that the scaling function for the external hulls below threshold is the same as the scaling function for the internal hulls above threshold. We computed the distribution exponent, $\tau' = 2.1425 \pm 0.0003$, as well as the fractal dimension of the hulls, $d_f^S = 1.750 \pm 0.001$. Finally, we argued that the critical exponent σ' might be the same as for regular percolation. This was confirmed numerically by computing the scaling functions. We also constructed the reduced hulls for this problem and determined their fractal dimension, $d_f^R = 1.334 \pm 0.004$. All these results strongly suggest that the surface critical exponents of antipercolation and percolation clusters are equal, as is the case for the bulk exponents.

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